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HEREDITARY MODULAR GRAPHS

H.-J. BANDELT

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In a hereditary modular graph G, for any three vertices u, v, w of an isometric subgraph of G, there exists a vertex of this subgraph that is simultaneously on some shortest u, v-path, u, w-path and v, w-path. It is shown that the hereditary modular graphs are precisely those bipartite graphs which do not contain any isometric cycle of length greater than four. There is a polynomial-time algorithm available which decides whether a given (bipartite) graph is hereditary modular or not. Finally, the chordal bipartite graphs are characterized by forbidden isometric subgraphs.

A property of graphs is usually said to be hereditary if it is inherited by all induced subgraphs. If a property is not hereditary, then one can turn it into a hereditary one by requiring it for all induced subgraphs. In some instances one can characterize a hereditary class of graphs by a "small" list of forbidden induced subgraphs. A prime example of such a situation is provided by the following result of Golumbic and Goss [4]: all induced subgraphs of a finite bipartite graph G admit a perfect elimination scheme if and only if G has no induced cycles of length greater than four. Bipartite graphs all of whose induced cycles have length four are called *chordal.* That kind of approach, however, does not apply to metric features of graphs, i.e., ones involving the distance function d. In this case one is better off with properties inherited by all isometric subgraphs (rather than all induced subgraphs). An *isometric* subgraph of a graph G is an induced subgraph in which the distance of any two vertices x and y equals the distance $d(x, y)$ taken in G. The property we will consider here is "modularity", shared, for instance, by median graphs ([1]) and chordal bipartite graphs. A graph G is *modular* (sensu Howorka [7]) if for any three vertices u, v, w there exists a vertex which is on three shortest paths joining u and v, u and w, v and w, respectively. It is convenient to express this in terms of the interval function: the *interval* $I(u, v)$ between two vertices u and v consists of all vertices that are on shortest paths joining u and v, that is: a vertex w belongs to $I(u, v)$ if and only if $d(u, v)=d(u, w)+d(v, w)$ (cf. Mulder [9]). Then a graph is modular if and only if the intersections $I(u, v)\cap I(u, w)\cap I(v, w)$ are never empty. Necessarily, a modular graph is bipartite. The following observation is basic to Theorem 1.

Lemma ([1]). *A bipartite graph* G *is modular if and only if* $I(u, v)\cap I(u, w)\cap I(v, w)$ *is non-empty for each triple* u, v, w *of vertices such that* $d(v, w)=2$ *and* $d(u, v)=d(u, w)\geqq 2$.

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Proof. Suppose that G is not modular but satisfies the condition of the lemma. Choose a triple u, v, w violating modularity and a shortest v, w-path P such that the distance sum of u to the vertices of P is as small as possible. If there is a subpath $x \to y \to z$ of P with $d(u, x) = d(u, y) - 1 = d(u, z)$, then there exists a vertex y' in $I(u, x) \cap I(u, z) \cap I(x, z)$. Then replacing y by y' results in a path with a smaller distance sum to u than P, thus contradicting minimality. Therefore, assuming that $d(u, v) \leqq d(u, w)$, the set $\{d(u, p) | p$ is a vertex on $P\}$ has either a unique maximum (viz.: $d(u, v) + d(v, w)$) or a unique minimum less than $d(u, v)$. In either case, the intersection of $I(u, v)$, $I(u, w)$, and $I(v, w)$ would be non-empty, contrary to the choice of u, v, w. This finishes the proof. ∎

Now, a graph is called *hereditary modular* if every isometric subgraph is modular. Observe that the graphs of Fig. 1 and 2 are modular but not hereditary modular. Indeed, either graph contains an induced/isometric cycle of length six, which is not a modular graph in its own right. The absence of isometric cycles of length greater than four is actually characteristic for hereditary modular graphs, see Theorem 1. Based on this it is easy to give a recursive characterization of finite hereditary modular graphs, see Theorem 2. In the concluding Proposition the chordal bipartite graphs are characterized as those hereditary modular graphs which do not contain any isometric subgraph of radius two with an induced cycle of length greater than six. Finally, instances of chordal bipartite graphs are bipartite distance-hereditary graphs and multitrees.

All graphs considered here are connected (and without loops, of course), but not necessarily finite (unless stated otherwise).

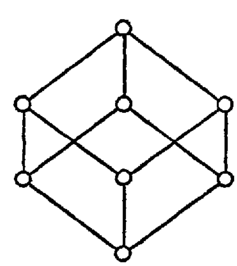


Fig. 1

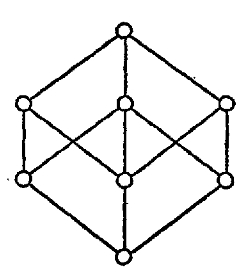


Fig. 2

Theorem 1. *The following conditions are equivalent for any (bipartite) graph G:*
(i) *G is a hereditary modular graph,*
(ii) *G is a modular graph such that the graphs of Fig. 1, 2 are not induced subgraphs of G,*
(iii) *G is a modular graph without induced cycle of length six,*
(iv) *every isometric cycle in G has length four.*

Proof. (i) implies (ii): This is obvious.

(ii) implies (iii): Let C be an induced cycle of length six in G. Then for either triple of non-adjacent vertices on C there is a common neighbour in G. If these two neighbours are not adjacent, then we get an induced cube (the graph of Fig. 1), and otherwise, we get the graph of Fig. 2 as an induced subgraph of G.

(iii) implies (iv): This follows from the fact that if a modular graph contains an isometric cycle of length $2n \geqq 8$, then it contains an isometric cycle of length $2n-2$. To verify this, let $u_1 \to u_2 \to u_3 \to \ldots \to u_{2n} \to u_1$ be an isometric cycle of length $2n \geqq 8$ in G. By modularity there exists a vertex v_{n+1} of G adjacent to u_n and u_{n+2} such that $d(u_1, v_{n+1}) = n-2$. Then we can choose a common neighbour v_{n+2} of v_{n+1} and u_{n+3} at distance $n-3$ from u_1. In a similar fashion we then choose vertices v_{n+3}, v_{n+4}, and so on, until at the last step a common neighbour v_{2n-2} of v_{2n-3}, u_{2n-1}, and u_1 is chosen. For each $1 \leqq i \leqq n-2$ the vertices u_{i+1} and v_{n+i} are at distance $n-1$. This is so because each u_{n+i+1} is adjacent to v_{n+i} and at distance n from u_{i+1}. Therefore the vertices $u_1, u_2, \ldots, u_n, v_{n+1}, v_{n+2}, \ldots, v_{2n-2}$ induce an isometric cycle of length $2n-2$ (see Fig. 3).

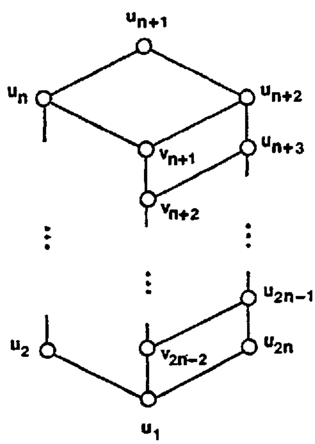


Fig. 3

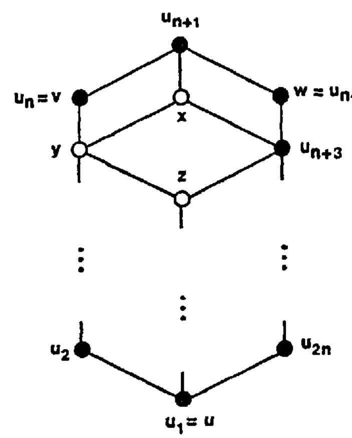


Fig. 4

(iv) implies (i): Suppose by way of contradiction that G is not modular. Choose vertices u, v, w violating modularity, where $d(v, w) = 2$, and $n = d(u, v) + 1 = d(u, w) + 1 \geqq 3$ is as small as possible. Thus, v and w have a common neighbour u_{n+1} at distance n from u, but $I(u, v) \cap I(u, w) = \{u\}$. Now let $u = u_1 \to u_2 \to \ldots \to u_{n-1} \to u_n = v$ and $w = u_{n+2} \to u_{n+3} \to \ldots \to u_{2n} \to u_1 = u$ be any shortest paths joining u and v, w and u, respectively (see Fig. 4). We assert that for each $i = 1, \ldots, n$ the following holds (set $u_{2n+1} = u_1$):

$$(*) \qquad d(u_i, u_{n+i}) = n \quad \text{and} \quad I(u_i, u_{n+i-1}) \cap I(u_i, u_{n+i+1}) = \{u_i\}.$$

For $i=1$ this is true by the choice of u, v, w. Then the cycle $u_1 \to \ldots \to u_{2n} \to u_1$ is certainly induced. Since isometric cycles of length greater than four are forbidden, we infer that $n \geqq 4$. Further, $d(u_2, u_{n+2})=n$ holds because u_2 cannot belong to the interval $I(u_1, u_{n+2})$. Now, as u_{n+1}, u_{n+3} are in $I(u_2, u_{n+2})$, we infer from the minimality assumption that either the intervals $I(u_2, u_{n+1})$ and $I(u_2, u_{n+3})$ have only the vertex u_2 in common or there exists a common neighbour x of u_{n+1} and u_{n+3} at distance $n-2$ from u_2. Assume the latter: then u_n and x belong to $I(u_2, u_{n+1})$, and hence, again by minimality, there is a common neighbour y of u_n and x with $d(u_2, y)=n-3 \geqq 1$, see Fig. 4. Then u_{n+3} and y are adjacent to x and at distance $n-2$ from u_1, whence there must be some common neighbour z of u_{n+3} and y such that $d(u_1, z)=n-3$ and $d(u_{n+1}, z)=3$. This, however, conflicts with $n \geqq 4$. We conclude that $(*)$ is true for $i=2$. Continue in a similar fashion, until at the last step $(*)$ is verified for $i=n$. Since for each i the distance of u_i and u_{n+i} equals n, the vertices $u_1, \ldots, u_{2n}$ induce an isometric cycle of length greater than six, arriving at a final contradiction. Therefore, by the Lemma, the graph G must be modular. For every isometric subgraph of G the same conclusion holds, whence G is hereditary modular, as required. ∎

Theorem 1 can be used to improve the corollary to Proposition 3 of Hell [5] concerning absolute retracts. By definition, an n-chromatic graph G is an *absolute retract* if every isometric embedding of G in an n-chromatic graph H is a coretraction, that is, there exists an idempotent edge-preserving map from H onto G. A graph G is called a *hereditary absolute retract* if every isometric n-chromatic subgraph of G is an absolute retract. It is quite easy to see that a bipartite absolute retract G is modular (cf. [2] Theorem 4.2): given three vertices u_1, u_2, u_3 in G such that the intersection of all $I(u_i, u_{i+1})$ (where indices are taken modulo 3) is empty, add a new vertex x to G and new u_i, x-paths P_i $(i=1, 2, 3)$ of length

$$\frac{1}{2}(d(u_i, u_{i+1})+d(u_i, u_{i+2})-d(u_{i+1}, u_{i+2}));$$

then the resulting graph extends G isometrically, and any retraction onto G maps x to a vertex of G belonging to all three intervals $I(u_i, u_{i+1})$ $(i=1, 2, 3)$, contrary to the assumption. On the other hand, if G is a finite hereditary modular graph, then from [2] Theorem 4.2 (part (i)$\Leftrightarrow$(ii)) one can deduce that G is an absolute retract. Alternatively, this implication can be proven by applying Theorem 3.5 of Pesch and Poguntke [10], which guarantees that a finite bipartite graph G is an absolute retract whenever for each diametrical vertex z the vertex-deleted subgraph $G-z$ is a retract of G and an absolute retract. Now, from Theorem 2 and its proof below one can derive that deleting a diametrical vertex from a finite hereditary modular graph results in a retract, which is then hereditary modular and hence an absolute retract by an inductive hypothesis. We therefore obtain from Theorem 1 the following result.

Corollary 1. *A finite bipartite graph is a hereditary absolute retract if and only if it is hereditary modular.*

Now, in a finite bipartite graph G isometric cycles of length greater than four are readily detected by repeatedly removing extremal vertices, as the next theorem confirms. A vertex z is called *extremal* if, for some vertex u, no interval $I(u, v)$ contains $I(u, z)$ properly, that is (as G is bipartite), if the neighbourhood $N(z)$ (consisting of all neighbours of z) is contained in $I(u, z)$. Note that in any finite bipartite graph there exist extremal vertices: for each vertex u we may choose a vertex z at maximal distance from u; then z is necessarily extremal. We then get the following recursive characterization of finite hereditary modular graphs.

Theorem 2. *For a finite bipartite graph G with at least three vertices, the following conditions are equivalent:*
(i) *G is hereditary modular,*
(ii) *for every extremal vertex z of G, the vertex-deleted subgraph $G-z$ is isometric in G and hereditary modular, and for any vertices u, v with*

$$d(u, v) = d(u, z) = d(v, z) = 2$$

at least one of the following inclusions holds:

$$N(u)\cap N(v) \subseteqq N(z),$$

$$N(u)\cap N(z) \subseteqq N(v),$$

$$N(v)\cap N(z) \subseteqq N(u),$$

(iii) *there exist two vertices y and z of G with $N(z)\subseteqq N(y)$ such that the vertex-deleted subgraph $G-z$ is hereditary modular, and for $x\in N(y)-N(z)$ and $u, v\in N(x)$ with $d(u, z)=d(v, z)=2$, either $N(u)\cap N(z)\subseteqq N(v)$ or $N(v)\cap N(z)\subseteqq N(u)$ holds.*

Proof. (i) implies (ii): Let z be an extremal vertex of G. Then all neighbours of z belong to some interval $I(u, z)$. Since any two neighbours of z have a common neighbour in $I(u, z)$ different from z by virtue of modularity, $G-z$ is an isometric subgraph of G. Therefore $G-z$ has only cycles of length four, and hence is hereditary modular by Theorem 1. Suppose that for some vertices u and v the three pairwise intersections of the sets $N(u), N(v), N(z)$ are not empty, but are incomparable. Then select a vertex from each intersection not belonging to the other intersections. Now these three vertices together with u, v, and z induce a 6-cycle in G, which is forbidden. This proves (ii).

(ii) implies (iii): First we show that for any extremal vertex z there exists another vertex y whose neighbourhood contains $N(z)$. If z has degree one, then there is nothing to show. So let z have at least two neighbours. Pick a vertex u such that $N(z)$ is contained in $I(u, z)$. Let y be a vertex in $I(u, z)$ such that $N(y)\cap N(z)$ has a maximum number of vertices. This number is at least two because $G-z$ is isometric in G. We assert that $N(z)$ is a subset of $N(y)$. Suppose it is not: then choose any vertex $t\in N(z)-N(y)$ and a vertex $x\in N(t)$ such that $N(x)\cap N(y)\cap \cap N(z)$ has maximum cardinality. Notice that the latter set is not empty since $G-z$ is isometric in G. By maximality of $N(y)\cap N(z)$, there exists a common neighbour r of y and z not adjacent to x. Again by isometry, we can find a common neighbour w of r and t in $G-z$ (necessarily different from x and y). Now x, t, w, r, y with each vertex s from $N(x)\cap N(y)\cap N(z)$ induce a 6-cycle in $G-z$ unless s is adjacent to w. Since induced 6-cycles cannot occur in the hereditary modular graph $G-z$, the neighbourhood of w contains $N(x)\cap N(y)\cap N(z)$. This, however, contradicts maximality of the latter set as $r \in N(w) \cap N(y) \cap N(z)$. We conclude that $N(z)$ is indeed a subset of $N(y)$.

Finally, if u, v, x are three vertices such that $x \in N(y) - N(z)$ and u, $v \in N(x)$ with $d(u, z) = d(v, z) = 2$, then, in particular, $d(u, v) = 2$ and $N(u) \cap N(v) \subsetneqq N(z)$, whence the conclusion follows from condition (ii).

(iii) implies (i): Let condition (iii) be fulfilled for some vertices y and z. Suppose that G contains an isometric cycle $C = u_1 \to u_2 \to \ldots \to u_{2n} \to u_1$ of length $2n \geqq 6$. Necessarily, the vertex z lies on this circle, whence $u_{n+1} = z$, say. Then y is at distance $n-2$ from u_1, for otherwise, we could substitute z by y and thus get an isometric cycle in $G - z$. If $n = 3$, then the vertices $u = u_2$, $v = u_6$, and $x = u_1$ violate condition (iii). So we must have $n \geqq 4$. Put $v_{n+1} = y$, and choose vertices $v_{n+2}, \ldots, v_{2n-2}$ as in the proof of Theorem 1, see Fig. 3; here the existence of the vertices v_i is guaranteed by modularity of $G - z$. Now we have an isometric cycle $u_1 \to \ldots \to u_n \to v_{n+1} \to \to \ldots \to v_{2n-2} \to u_1$ of length greater than four, which does not contain z, giving a contradiction. We conclude that G is hereditary modular. This completes the proof of Theorem 2. ∎

Theorem 2 suggests the following algorithm which decides whether a graph is hereditary modular or not: the input is a finite bipartite graph given by its list of neighbourhoods. If G has fewer than six vertices, then G is hereditary modular. Otherwise, find vertices y, z with $N(z) \subseteqq N(y)$ (if there are none, G is not hereditary modular). If there exist $x \in N(y) - N(z)$ and u, $v \in N(x)$ such that $N(u) \cap N(z) \subsetneqq N(v)$ and $N(v) \cap N(z) \subsetneqq N(u)$, then G is not hereditary modular. Otherwise, delete z and start again with $G - z$ instead of G. Reversing this procedure gives a method of construction, which amounts to succesively adjoining vertices z and giving them neighbours from already constructed single neighbourhoods $N(y)$, where the only matter of concern is that no induced 6-cycles are created.

A particular instance of Theorem 2 is worth mentioning. In view of Theorem 1 and a result from [1], the hereditary modular graphs without induced $K_{2,3}$ are precisely the hereditary median graphs (viz., the cubefree median graphs = median graphs without induced cube). Equally, this fact can be obtained from Theorem 1 in conjunction with [9] Theorem 3.1.8 and the elementary fact that an induced 6-cycle in a median graph gives rise to an induced cube. Recall that a *median* graph is a (bipartite) graph for which all intersections $I(u, v) \cap I(u, w) \cap I(v, w)$ are singletons. It is easy to see that extremal vertices in a cubefree median graph G have degree at most two. Every vertex of degree two in G is either an extremal vertex or a cut vertex such that the corresponding vertex-deleted subgraph has exactly two (cubefree median) components. If, conversely, a vertex-deleted subgraph $G - z$ is

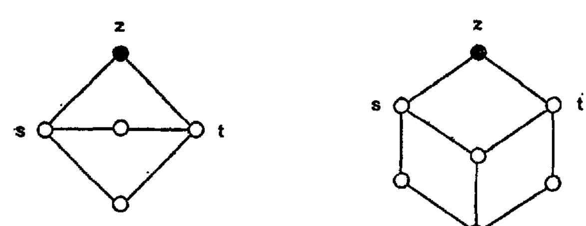


Fig. 5

a cubefree median graph for some vertex z of degree two in a bipartite graph G, then G is a cubefree median graph provided that in G neither instance of Fig. 5 occurs.

Corollary 2. *A finite bipartite graph G (with at least three vertices) is a cubefree median graph if and only if either* (1) *there exists a vertex z of degree one such that $G-z$ is a cubefree median graph, or* (2) *there exists a vertex z of degree two such that $G-z$ is a cubefree median graph and the two neighbours s and t of z have exactly one common neighbour $y \neq z$, and moreover, every neighbour $u \neq y$ of s is at distance four to any neighbour $v \neq y$ of t.*

Another important subclass of hereditary modular graphs is formed by the chordal bipartite graphs. In this context it is of interest to characterize these graphs by forbidden isometric subgraphs. This is in fact easy to accomplish: the forbidden graphs in question are (besides long cycles) the bipartite analogues of wheels.

Proposition. *A hereditary modular graph G is a chordal bipartite graph if and only if no graph from the infinite lists of Fig.* 6 *(bipartite wheels) and Fig.* 7 *(bipartite double-wheels) is an isometric subgraph of G.*

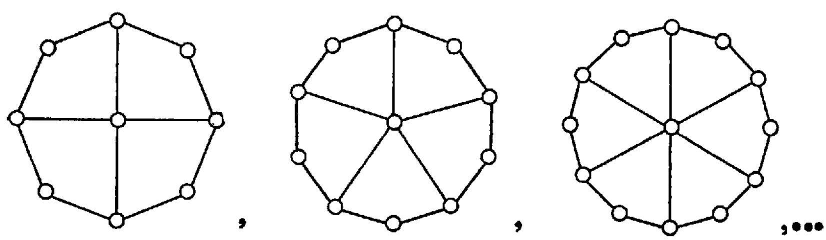


Fig. 6

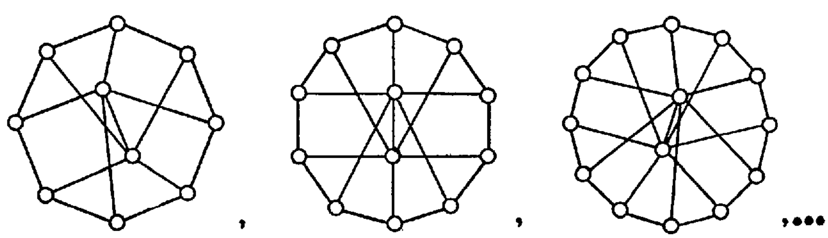


Fig. 7

Proof. Each graph of Fig. 6, 7 has an induced cycle of length at least eight, and hence cannot occur isometrically in a chordal bipartite graph.

Conversely, assume that the hereditary modular graph G has some induced cycle $C=u_1 \to \ldots \to u_{2n} \to u_1$ of length $2n \geqq 6$, where n is minimal. Necessarily, this cycle is not isometric in G (whence $n \geqq 4$). So, without loss of generality we may assume that $k=d(u_1, u_m) \leqq m-3$ for some $5 \leqq m \leqq n+1$, where k is as small as possible. Pick any shortest path P joining u_1 and u_m. Since k is minimal, P intersects C only in u_1 and u_m. Note that, by minimality of n, the two cycles formed by P together with $u_1 \to \ldots \to u_m$ and $u_m \to \ldots \to u_{2n} \to u_1$, respectively, are not induced. Now let v be the neighbour of u_1 on P. First suppose that u_1 is the unique neighbour of

v on C (whence $k \geqq 3$). Then there exists a vertex w on $P-u_m$ adjacent to some u_i with $2 \leqq i < m$ such that $d(u_1, w)$ is minimal. If w is not adjacent to any vertex u_j with $2 \leqq j < i$, then $u_1 \to \ldots \to u_i \to w$ and the subpath of P joining w and u_1 give an induced cycle of length greater than four but less than $2n$, which is impossible. Therefore w must be adjacent to both u_2 and v. Similarly we infer that w is adjacent to u_{2n}. Now, the only possible chords in the cycle $w \to u_2 \to \ldots \to u_{2n} \to w$ are of the form $u_{2j}w$. All of them are actually present, for otherwise, minimality of n or k would be violated. Then, in particular, we have $d(u_4, u_{2n}) = 2 < k$, contradicting minimality of k.

We conclude that v is adjacent to some u_{2j-1}, where $j \geqq 2$ is chosen as small as possible. Then as $v \to u_1 \to \ldots \to u_{2j-1} \to v$ is an induced cycle we get $j=2$ by minimality of n and k. Similarly as above it follows that in the cycle $v \to u_3 \to \ldots \to u_{2n} \to u_1 \to v$ all chords $u_{2i}v$ must exist. This gives an induced bipartite wheel with $n \geqq 4$ spokes. We assert that this bipartite wheel either is isometric in G or can be extended to some induced/isometric bipartite double-wheel. Suppose that the former is not true: then there exists a common neighbour w in G of two vertices u_{2i} and u_{2j} on C whose distance in the cycle C is at least four. Then as there is no induced 6-cycle in G the vertices v and w are adjacent. If w were not adjacent to all vertices of C with even index, then we would get an induced cycle passing through w whose length were greater than four but less than $2n$, contrary to minimality of n. Therefore the vertices v and w together with C induce a bipartite double-wheel, as required. Thus, in any case, G contains some graph of Fig. 6, 7 as an isometric subgraph. ∎

Examples of chordal bipartite graphs are provided by the bipartite distance-hereditary graphs. A *distance-hereditary* graph (sensu Howorka [6]) is a graph all of whose induced paths are isometric. In [3] it is shown that the bipartite distance-hereditary graphs are precisely those modular graphs which do not contain the graph of Fig. 8 as an induced/isometric subgraph. To see this directly, first observe that a bipartite distance-hereditary graph cannot have an induced subgraph of the latter kind and must be modular in view of Theorem 1. Conversely, consider a modular graph G which is not distance-hereditary, so that there exists an induced path $u_0 \to u_1 \to \ldots \to u_m$ with $m > d(u_0, u_m)$ being as small as possible. Let $k < m$ be the largest index such that the subpath from u_0 to u_k is isometric. Then, by modularity, u_{k-1} and u_{k+1} have a common neighbour u_k' with $d(u_0, u_k') = k-2$. Then u_k' must be adjacent to u_{k-3} since the path $u_0 \to \ldots \to u_{k-1} \to u_k'$ cannot be induced by minimality of m. Thus, $u_k', u_{k-3}, \ldots, u_{k+1}$ induce the graph of Fig. 8.

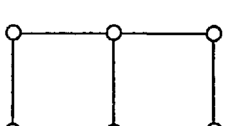


Fig. 8

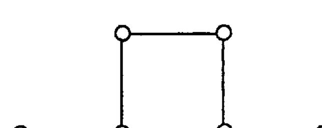
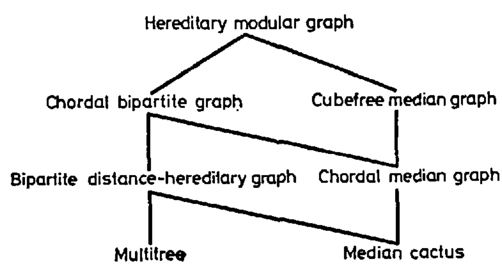


Fig. 9

Multitrees constitute a still more restrictive class of hereditary modular graphs. By a *multitree* we mean the covering graph of any discrete multitree poset. Multitree posets have been investigated by Jung [8]: their comparability graphs are precisely the distance-hereditary graphs of diameter at most two. Every multitree is obtained from a tree by splitting vertices. In other words, a bipartite graph G is a multitree if and only if any maximal subgraph of G containing no two vertices with identical neighbourhoods is a tree. Certainly, multitrees must be distance-hereditary and cannot have the graph of Fig. 9 as an induced subgraph. Conversely, a bipartite graph which is not a multitree contains either an induced cycle of length greater than four or a 4-cycle $u \to v \to w \to x \to u$ with $N(u) \neq N(w)$ and $N(v) \neq \neq N(x)$ (and thus contains the graph of either Fig. 8 or Fig. 9).

As was mentioned above cubefree median graphs are hereditary modular. Observe that a chordal median graph is a cubefree median graph without any isometric bipartite wheel, and a median cactus is a graph all of whose non-trivial blocks are 4-cycles.

Summarizing we have the following hierarchy of hereditary modular graphs:



References

[1] H. J. Bandelt, Characterizing median graphs, *European J. Combinatorics, to appear.*
[2] H. J. Bandelt, A. Dählmann and H. Schütte, Absolute retracts of bipartite graphs, *Discrete Applied Math.*, **16** (1987), 191—215.
[3] H. J. Bandelt and H. M. Mulder, Distance-hereditary graphs, *J. Combinatorial Theory Ser. B* **41** (1986), 182—208.
[4] M. C. Golumbic and C. F. Goss, Perfect elimination and chordal bipartite graphs, *J. Graph Theory* **2** (1978), 155—163.
[5] P. Hell, Absolute retracts in graphs, *Lecture Notes Math.* **406** (1974), 291—301.
[6] E. Howorka, A characterization of distance-hereditary graphs, *Quart. J. Math. Oxford Ser. 2* **28** (1977), 417—420.
[7] E. Howorka, Betweenness in graphs, *Abstracts Amer. Math. Soc.* **2** (1981), *783—06—5;
[8] H. A. Jung, On a class of posets and the corresponding comparability graphs, *J. Combinatorial Theory Ser. B* **24** (1978), 125—133.
[9] H. M. Mulder, The interval function of a graph, *Math. Centre Tracts* **132** (Mathematisch Centrum, Amsterdam, 1980).
[10] E. Pesch and W. Poguntke, A characterization of absolute retracts of n-chromatic graphs, *Discrete Math.* **57** (1985), 99—104.

H.-J. Bandelt

Fachbereich 6
Universität Oldenburg
D-2900 Oldenburg, BRD